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ON SYSTEMS OF SIX POINTS LYING IN THREE WAYS IN INVOLUTION.

By PROF. H. MASCHKE, Chicago, Ill.

In the following paper a system of 6 points will be studied which are in involution in three ways such that no pair of conjugate points occurs more than once. Sextuples of points of this kind are of importance for several mathematical problems. They not only occur in geometrical investigations, as for instance in the theory of multiply perspective triangles, in Clebsch's hexagon, in the metharmonic division, etc.,* but also in analytic researches.† As shown in § 1, the theory of these points is identical with the theory of the dihedron for $n = 3$. In most of the above mentioned problems those cases are of peculiar interest in which some of the anharmonic ratios that can be formed out of 4 points contained in the sextuple, have real values. To these cases the greater part of the present paper is devoted (§§ 2–5); in particular, the case in which all the 6 points are real is studied in § 5.

§ 1. THE GENERAL THEORY.

If 6 points, defined in the plane of complex numbers by the values of a complex variable z, are in involution, there exists a linear transformation of z of period 2 producing the required correspondence of the 6 points in conjugate pairs.

If the 6 points are twice in involution in such a way that the two involutions have no pair of conjugate points in common, then the notation of the 6 points 1, 2, ..., 6 can always be arranged so that the first involution can be written

$$S : (14)\,(26)\,(35)\,, \tag{1}$$

the second

$$T : (15)\,(24)\,(36)\,. \tag{2}$$

Denoting the linear z-transformations corresponding to (1) and (2) by S and

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* Besides the references given below in this paper, cf. Rosanes: Ueber Dreiecke in perspectivischer Lage. Math. Annalen, Bd. 2, p. 549; Schröter: Ueber perspectivisch liegende Dreiecke. Math. Annalen, Bd. 2, p. 553; Fuortes: Dimonstratione di due theorimi di geometria. Battaglini G. IX, p. 50; E. H. Moore: A problem suggested in the geometry of nets of curves applied to the theory of 6 points having multiply perspective relations. American Journ. Vol. 10, p. 241.

† cf. Hutchinson: On the reduction of hyperelliptic functions to elliptic functions by a transformation of the second degree. Thesis, University of Chicago, 1895.

T, respectively, we find that the transformation STS, also of period 2, produces the following involution :

$$STS : (16)\,(25)\,(34)\,. \tag{3}$$

Hence we have the theorem,

If 6 *points are twice in involution in such a way that the two involutions have no pair of conjugate points in common, then they are always a third time in involution.*

When S and T are combined in all possible manners, only 6 distinct transformations are obtained. Besides the identity and (1), (2), (3), we shall have the two following :—

$$ST : (123)\,(456)\,, \tag{4}$$

$$(ST)^2 : (132)\,(465)\,. \tag{5}$$

These 6 transformations as well as the 6 corresponding permutations of the 6 points form a group, the well known dihedron-group for $n = 3$.

Let $z_1, z_2, \ldots, z_6$ be the values of z in the 6 points 1, 2, . . . , 6, and

$$\varphi(z) = (z - z_1)\,(z - z_2) \ldots (z - z_6)\,; \tag{6}$$

then $\varphi(z)$ admits the linear transformations of a dihedron-group, $n = 3$, i. e. if the 6 linear, homogeneous substitutions of a certain dihedron-group are applied to z_1 and z_2, $\varphi(z)$ remains unaltered, when written in the homogeneous form by putting $z = z_1 : z_2$. We will therefore call these points *six dihedron-points.*

We now describe a sphere with radius 1 and with the zero-point O of the z-plane as center, and determine a point P on this sphere by rectangular coordinates ξ, η, ζ, where the plane $\zeta = 0$ coincides with the z-plane, and the ξ-axis with the axis of real numbers. If then N denotes the point of intersection between the sphere and the positive direction of the ζ-axis, θ the angle NOP, and φ the angle between the plane NOP and the ξ-ζ plane, we have

$$\left.\begin{aligned} \xi &= \sin\theta\cos\varphi\,, \\ \eta &= \sin\theta\sin\varphi\,, \\ \zeta &= \cos\theta\,; \end{aligned}\right\} \tag{7}$$

and the projection of P from N on the z-plane is determined by

$$z = \frac{\xi + i\eta}{1 - \zeta}^{*} = \frac{e^{i\phi} \,.\, \sin\theta}{1 - \cos\theta}\,,$$

or

$$z = e^{i\phi} \,.\, \cot \tfrac{1}{2}\,\theta\,. \tag{8}$$

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* cf. Klein, Ikosaëder, p. 32.

Let now the sphere undergo the rotations of a dihedron-group, $n = 3$. An arbitrary point a_1 on this sphere may be transferred to a_2 and a_3 by a rotation of $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$ about the ζ-axis. Let a rotation of π about the ξ-axis transfer a_1 to a_4, a_2 to a_6 and a_3 to a_5. The projections of these 6 points from N are then defined by the following values of z which we will denote by capital letters :

$$\left.\begin{array}{ll} Z_1 = e^{i\phi} \cot \tfrac{1}{2}\,\theta\,, & Z_4 = e^{-i\phi} \tan \tfrac{1}{2}\,\theta\,, \\ Z_2 = \varepsilon e^{i\phi} \cot \tfrac{1}{2}\,\theta\,, & Z_5 = \varepsilon e^{-i\phi} \tan \tfrac{1}{2}\,\theta\,, \\ Z_3 = \varepsilon^2 e^{i\phi} \cot \tfrac{1}{2}\,\theta\,, & Z_6 = \varepsilon^2 e^{-i\phi} \tan \tfrac{1}{2}\,\theta\,, \end{array}\right\} \tag{9}$$

where $\varepsilon = e^{\frac{2}{3}\pi i}$.

Putting now

$$\Phi(z) = (z - Z_1)(z - Z_2) \ldots (z - Z_6)\,, \tag{10}$$

we find

$$\Phi(z) = z^6 - mz^3 + 1\,, \tag{11}$$

where

$$m = e^{3i\phi} \cot^3 \tfrac{1}{2}\,\theta + e^{-3i\phi} \tan^3 \tfrac{1}{2}\,\theta\,. \tag{12}$$

To this form Φ every binary sextic which remains unchanged for a dihedron-group, $n = 3$, must be reducible by a linear transformation.*

The configuration of the 6 points $Z_1, Z_2, \ldots, Z_6$ (9) consists of 2 regular triangles with the common center O; it will be called the *normal-position* of 6 dihedron-points.

We have now the theorem,

If 6 *points are twice in involution in such a way that the two involutions have no pair of conjugate points in common, then they form a sextuple of dihedron-points and can always be transformed by a linear substitution into the normal-position of* 6 *dihedron-points.*

Those 6 linear transformations which convert the 6 dihedron-points of the normal-position into themselves, are given by

$$z' = z\,, \quad z' = \varepsilon z\,, \quad z' = \varepsilon^2 z\,, \tag{13}$$

and

$$z' = 1/z\,, \quad z' = \varepsilon/z\,, \quad z' = \varepsilon^2/z\,. \tag{14}$$

The three latter are of period 2 and represent the three involutions (1), (2), and (3). The double points of these involutions are obtained by putting $z' = z$ in (14), which gives $z = \pm 1, \pm \varepsilon, \pm \varepsilon^2$. Hence

The double points of the 3 *involutions of* 6 *dihedron-points in the normal-position are entirely independent of the special position of the point* a_1 *on the sphere, i. e. of* θ, φ, *and* m *; they represent on the equator (intersection between sphere and z-plane) a regular hexagon :* $z^6 - 1 = 0$.

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* cf. Klein, Ikosaëder, p. 49, and Bolza : On binary sextics with linear transformations into themselves, American Journal, Vol. X, p. 49.

§ 2. ANHARMONIC RATIOS.

The anharmonic ratios of the quadruples of points contained in the dihedron-sextuple have, in general, complex values. Of special interest are those cases in which some of these ratios are real, and, in particular, harmonic.

The anharmonic ratio (12) (34) of 4 points z_1, z_2, z_3, z_4 in the complex plane is defined as

$$(12)\,(34) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}. \tag{15}$$

The necessary and sufficient condition for the reality of this anharmonic ratio is that the 4 points lie on a circle, and in this case the above defined anharmonic ratio is identical with the anharmonic ratio of 4 points on a circle (conic section) as defined in projective geometry.*

The projection of any circle of the sphere from N on the z-plane is again a circle, and a circle on the z-plane remains a circle if any linear transformation is applied.

We have therefore to find out those cases in which 4 of the 6 points a_1, $a_2, \ldots, a_6$ on the sphere lie on a circle (are concyclic). Unless all the 6 points lie on the equator, the circle passing through the 3 points a_1, a_2, a_3 on the upper hemisphere cannot contain any one of the 3 other points. A quadruple of concyclic points can therefore consist only of 2 points of the upper and 2 points of the lower hemisphere. An easy geometrical consideration gives then the 3 following possibilities for the existence of concyclic quadruples :—

1. The points 1, 2, 3, of the upper hemisphere lie perpendicularly (with respect to the equator) above the points 4, 5, 6 (in this order) of the lower hemisphere, corresponding to $\varphi = 0$, or $\varphi = \pi$ (θ arbitrary). The 6 points form a right prism.

2. The 2 triangles 123 and 456 lie symmetrically to each other, corresponding to $\varphi = \pm \frac{1}{6}\pi$, $\pm \frac{1}{2}\pi$, $\pm \frac{5}{6}\pi$ (θ arbitrary). The 6 points form an oblique octahedron.

3. All the 6 points lie on the equator, corresponding to $\theta = \frac{1}{2}\pi$ (φ arbitrary).

In the two cases, 1 and 2, the 3 quadruples (1245), (2356), (3164) lie each on a circle. The anharmonic ratios of these 3 quadruples taken in corresponding orders are evidently, as the figure on the sphere shows, equal, also in the general case without any specification of the values of φ and θ. Denoting

$$(15)\,(42) = (26)\,(53) = (34)\,(61) = \lambda\,, \tag{16}$$

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* cf. Klein : Einleitung in die geometrische Functionentheorie. Autographirte Vorlesung, 1880-81, pp. 42 and 43 With regard to the geometrical interpretation of anharmonic ratios cf. Moebius : Die Theorie der Kreisverwandtschaft. Crelles Journal, Bd. 52, or Gesammelte Werke, Bd. II ; and Wedekind : Beiträge zur geometrischen Interpretation binärer Formen. Math. Annalen, Bd. 9.

we find from (9) and (15)

$$\lambda = -\tfrac{1}{8}(e^{i\phi}\cot\tfrac{1}{2}\theta - e^{-i\phi}\tan\tfrac{1}{2}\theta)^2. \tag{17}$$

Eliminating $e^{i\phi}\cot\frac{1}{2}\theta$ between this equation and (12) we obtain

$$m^2 = (4 - 3\lambda)(1 - 3\lambda)^2, \tag{18}$$

or

$$\lambda(1-\lambda)^2 + \tfrac{1}{27}(m^2 - 4) = 0.$$

We now investigate the 3 cases pointed out above, in succession.

§ 3. THE 6 DIHEDRON-POINTS ON THE SPHERE FORM A RIGHT PRISM.

Since we have in this case $\varphi = 0$ or $\varphi = \pi$, it follows that

$$m = \pm(\cot^3\tfrac{1}{2}\theta + \tan^3\tfrac{1}{2}\theta). \tag{19}$$

The form $\Phi(z) = z^6 - mz^3 + 1$ represents therefore this case, if m takes any real value between $+2$ and $+\infty$ or between -2 and $-\infty$.

It is now sufficient to consider only the case $\varphi = 0$. We find from (17)

$$\lambda = -\tfrac{1}{8}(\cot\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\theta)^2,$$

which reduces to

$$\lambda = -\tfrac{4}{3}\cot^2\theta. \tag{20}$$

The 3 circles in which the 3 rectangular faces of the prism meet the sphere, intersect each other at equal angles. Denoting this angle by α, we have to find the connection between α, θ and λ.

The angle α remains unchanged by the projection from the sphere on the z-plane, and also by any linear transformation of the z-plane. To the normal-position representing our case (see Fig. 1) we apply a linear transformation

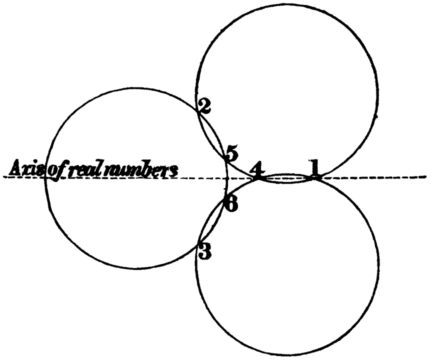

FIG. 1.

that converts the two circles (1452) and (1463) into straight lines, while the axis of real numbers remains unchanged. This is done by the transformation

$$z' = \frac{1}{z - Z_1}\,. \tag{21}$$

The point 1 is transferred to infinity and we obtain Fig. 2.

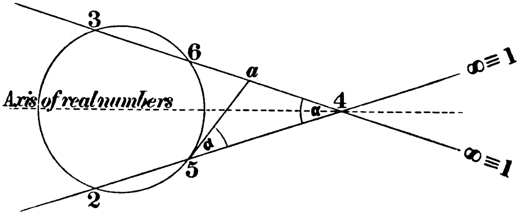

FIG. 2.

The connection between the anharmonic ratio λ (16) and the angle α can now be found in the following way :—

Draw the tangent to the circle in point 5 meeting the line 46 in point a. Then $\angle\, 45a = \alpha$. The anharmonic ratio of the pencil obtained by joining point 5 to the 4 points 2, 6, 5, 3 on the circle is also $= \lambda$. The intersection of the 4 rays of this pencil with the line 13 gives the points 4, 6, a, 3. Thus we have

$$\lambda = (46)\,(a3) = (34)\,(61)\,.$$

Denoting the distances $4a = r$, $46 = s$, $43 = t$, taken positive in the direction from 4, this double-equation becomes

$$\frac{r}{r - s} : \frac{t}{t - s} = \frac{s - t}{s} = \lambda\,,$$

and by elimination of t,

$$\frac{r}{s} = \frac{2 - \lambda}{1 - \lambda}\,.$$

On the other hand the distance 45 also $= s$, and triangle $5a4$ is isosceles ; whence $\cos \alpha = s/2r$.

Thus we obtain the required connection between α and λ,

$$\cos \alpha = \frac{2 - \lambda}{2 - 2\lambda}\,, \tag{22}$$

$$\lambda = \frac{2\,(\cos \alpha - 1)}{2 \cos \alpha - 1}\,. \tag{23}$$

From (20) and (23) we find

$$\cos a = \frac{3 - \cos^2 \theta}{3 + \cos^2 \theta}, \tag{24}$$

$$\cos \theta = \sqrt{3} \, . \tan \tfrac{1}{2} a \, . \tag{25}$$

Equation (24) shows that a varies only between 0 and $\frac{1}{3}\pi$, which means that the curvilinear triangle 456 lies always outside the 3 circles; while in the case $a > \frac{1}{3}\pi$, which is to be considered in § 4 (cf. Fig. 5), the triangle 456 lies inside each circle, the case $a = \frac{1}{3}\pi$ representing the limiting case, where the triangle is reduced to a point.

We have now obtained the following result:

The 6 *points of intersection of any* 3 *circles meeting each other at equal angles* a, *where* $a < \frac{1}{3}\pi$, *constitute* 6 *dihedron-points; they can always be linearly transformed into the normal-position obtained by projection of a right prism.*

If $\lambda = -1$, the 6 points are harmonically divided in the following remarkable manner:

(15) is divided harmonically by (24),
(26) . . . . (35),
(34) . . . . (16).

In this special case we have

$$\cos a = \tfrac{3}{4}, \quad \cos \theta = \sqrt{\tfrac{3}{7}}, \quad m = 4\sqrt{7} \, .$$

The rectangular faces of the right prism on the sphere are squares. It is this sextuple which I have called in a previous paper* a system of 6 *metharmonic* points. We see then that *the points of intersection of any* 3 *circles meeting at equal angles* a, *where cos* $a = \frac{3}{4}$, *form a system of metharmonic points.*

§ 4. THE 6 DIHEDRON-POINTS ON THE SPHERE FORM AN OBLIQUE OCTAHEDRON.

We have in this case $\varphi = \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{5}{6}\pi$, hence

$$m = \pm i \left(\cot^3 \tfrac{1}{2}\theta - \tan^3 \tfrac{1}{2}\theta\right) .$$

The form $\Phi(z) = z^6 - mz^3 + 1$ represents, therefore, this case, when m is purely imaginary.

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* H. Maschke: Ueber eine merkwürdige Configuration gerader Linien im Raume. Math. Annalen, Bd. 36, p. 200, and Nachrichten d. Königl. Gesellschaft d. Wissenschaften zu Göttingen, 1889, p 386.

Without loss of generality we may consider the case $\varphi = \frac{1}{2}\pi$ only. The position of the 6 points appears from Fig. 3 representing the perpendicular projection on the z-plane.

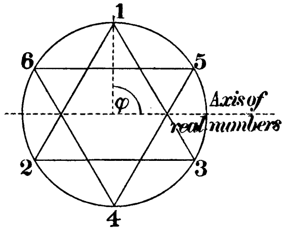

Fig. 3.

From (17) we find

$$\lambda = + \tfrac{1}{3} (\cot \tfrac{1}{2}\theta + \tan \tfrac{1}{2}\theta)^2 ,$$

or

$$\lambda = \tfrac{4}{3}/\sin^2 \theta \,. \tag{26}$$

Denoting, again, the common value of the angles at which the 3 circles containing the points (1245), (2356), (3164) meet each other by α, we obtain the connection between α and λ (16) in exactly the same way from Fig. 4, as the corresponding connection in § 3 had been found from Fig. 2. The formulæ (22) and (23) hold, therefore, in this case also. Furthermore, this figure is deduced by a linear transformation similar to (21) from the normal-position of the case presented in Fig. 5.

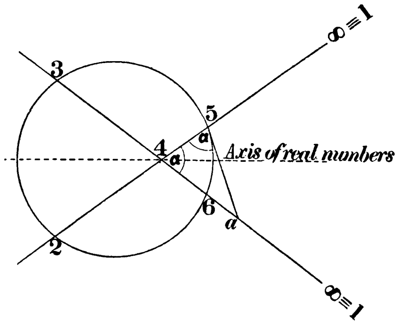

Fig. 4.

Comparing (23) and (26) we find

$$\cos \alpha = \frac{3 \cos^2 \theta - 1}{3 \cos^2 \theta + 1} ,$$

$$\cos \theta = \frac{1}{\sqrt{3}} \cot \tfrac{1}{2} \alpha \,.$$

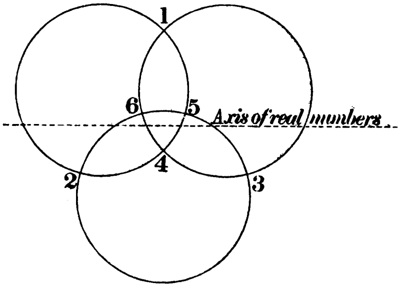

FIG. 5.

Consequently the angle α varies between $\frac{1}{3}\pi$ and π, and the anharmonic ratio λ between $+\frac{4}{3}$ and $+\infty$. Thus we have this result:—

The 6 *points of intersection of any* 3 *circles meeting each other at equal angles* α, *where* $\frac{1}{3}\pi < \alpha < \pi$, *constitute* 6 *dihedron-points; they can always be linearly transformed into the normal-position obtained by projection of an oblique octahedron.*

If $\lambda = 2$, the 6 points lie harmonically in such a way that every one of the 3 pairs (14), (25), (36) is harmonically divided by every other pair. In this special case we have

$$\alpha = \tfrac{1}{2}\pi, \quad \cos\theta = \frac{1}{\sqrt{3}}, \quad m = 5\sqrt{-2},$$

and the 6 points form on the sphere a *regular* octahedron.

§ 5. THE 6 DIHEDRON-POINTS OF THE SPHERE LIE ALL ON THE EQUATOR.

In this case we have $\theta = \frac{1}{2}\pi$ and therefore $m = 2\cos 3\varphi$. The form $\Phi(z) = z^6 - mz^3 + 1$ represents this case when m is real and varies between -2 and $+2$.

We transform the equator-circle into the axis of real numbers so that all the six points are real, and assume now, as we are only concerned with projective properties, the six points on a conic section (circle).

The lines joining the conjugate points of an involution meet in one point, the center of involution. Thus we obtain, corresponding to the 3 involutions (1), (2), (3) the 3 centers C_1, C_2, C_3 (see Fig. 6).

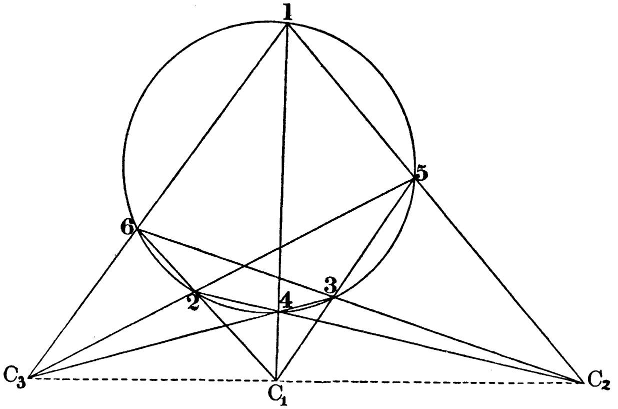

FIG. 6.

These 3 centers lie on one straight line, the Pascal-line of the hexagon 153426.

Furthermore, we see that the two triangles 123 and 456 are in perspective in 3 ways with C_1, C_2, C_3, as centers of perspective. Hess has studied in detail the case of two triply perspective triangles whose centers of perspective lie on a straight line*; he does not, however, mention that the 6 vertices of the triangles lie on a conic section, which is an immediate consequence of the converse of Pascal's theorem.

Varying the one parameter of the figure which is at our disposal—it depends on the value of φ in our normal-figure—by keeping for instance the 3 points 1, 4, 5 fixed and varying only the point 3 which determines the rest, we can find a position where the 3 lines (14), (25), (36) meet in one point. We then obtain a fourth involution (14), (25), (36), and the two triangles are in 4 ways in perspective. This case—two triangles that are perspective in 4 ways with 3 centers of perspective on a straight line—has also been studied by Hess.†

The corresponding normal-figure consists simply of a regular hexagon ($\varphi = \frac{1}{2}\pi$, $m = 0$, $\Phi(z) = z^6 + 1$); the additional center of involution is given by the center of the circle. This configuration is projectively unique; it represents the maximum number of involutions that can occur with 6 points, and also the maximum number of harmonic divisions, viz. six.

Reverting once more to Fig. 6 (φ arbitrary), let us draw tangents from C_1, C_2, C_3, to the circle. The 6 contact points represent the double points of the

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* Edmund Hess "Beiträge zur Theorie der mehrfach perspectiven Dreiecke und Tetraeder." Mathemat. Annalen, Bd. 28, pp. 186 et seq.

† l. c., pp. 188 and 198.

3 involutions (1), (2), (3), and as such they form in the normal-position a regular hexagon, as shown before. It follows, then, *that these* 6 *contact points lie in involution in* 4 *ways.*

We now ask how the 6 real dihedron-points on the conic section (circle) must lie in order that some of the anharmonic ratios may become harmonic. The normal-figure shows that, besides the case of the regular hexagon, this can only occur in such a way that (12) is divided harmonically by (45), (23) by (56), and (31) by (64); or so that (12) is divided by (56), etc. The latter case being essentially the same as the first, it suffices to consider the first only.

Putting in (17) $\theta = \frac{1}{2}\pi$, we obtain,

$$\lambda = (15)\,(42) = \tfrac{4}{3}\sin^2\varphi\,.$$

The value $\lambda = \frac{1}{2}$ gives harmonic division, whence

$$\cos\varphi = \tfrac{1}{4}\sqrt{10}\,,$$

and from (18)

$$m = \tfrac{1}{4}\sqrt{10}\,.$$

Let now the points 1, 2, . . . , 6 (see Fig. I) be any such system of real dihedron-points on a conic section. If we draw tangents in the points 1, 2, 3 to the conic section, determining the triangle A_1, A_2, A_3, then the lines joining the points 5–6, 6–4, 4–5, must meet the points A_1, A_2, A_3, respectively. Similarly the lines joining the points 2–3, 3–1, 1–2 must pass through the vertices A_4, A_5, A_6 of the triangle formed by the tangents in the points 4, 5, 6. Thus we are led to a known problem proposed by Clausen* and solved by Moebius†: *Given a triangle with an inscribed conic section* (*circle*). *To find a second triangle that is inscribed to the conic section* (*circle*) *and circumscribed to the given triangle.*

The configuration of this sextuple on the conic section is very closely connected with the configuration of a so-called *Clebschian hexagon.*‡ A Clebschian hexagon consists of 6 points C_1, C_2, . . . , C_6 (see Fig. II) situated in the plane in such a way that every one of the 10 pairs of triangles which can be formed out of the 6 points C, is in 4 ways in perspective. The 40 centers of perspective thus obtained are reduced to 10 (the points E_1, E_2, E_3, A_1, A_2, A_3, A_4, A_5, A_6 in Fig. II), since every one of them is a center of perspective for 4

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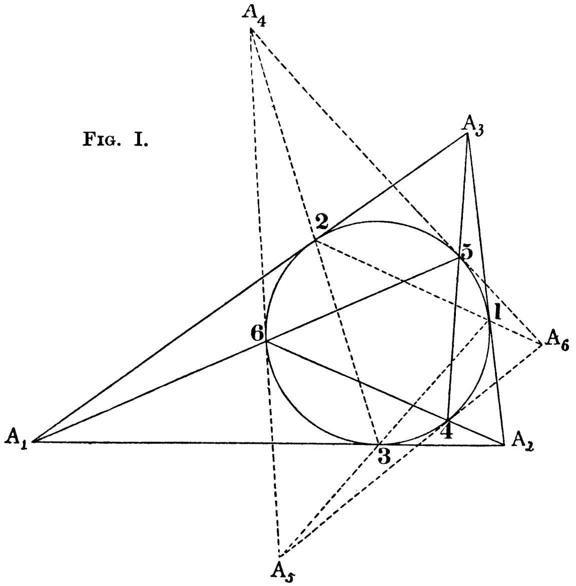
[*] Crelles Journ., Bd. 4, p. 391.

[†] Moebius: Barycentrische Lösung der Aufgabe des Herrn Clausen. Crelle's Journ., Bd. 5, and Gesammelte Werke, Bd. I, p. 487.

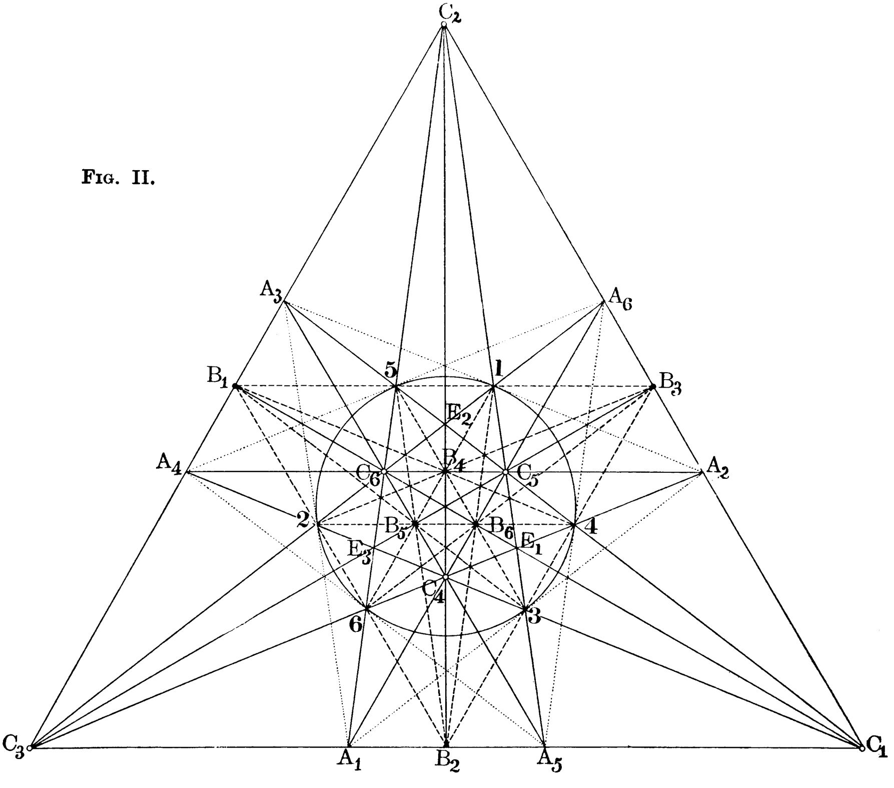
[‡] Clebsch. Math. Annalen, Bd. 4, pp. 284 and 345. This hexagon was more fully investigated by Schröter: Das Clebsch'sche Sechseck. Math. Annalen, Bd. 28, from which paper the above given principal properties are taken, and by Hess, l. c. § 6 under the name: Das zehnfach Brianchon'sche Sechseck. Cf. also: Klein, Ikosaeder, pp. 216–218, and Lindemann, Vorlesungen über Geometrie, II, 578.

Fig. I.



A_4
A_3
2
5
1
A_6
6
A_1
4
3
A_2
A_5

Fig. II.



C_2
A_3
A_6
5
1
B_1
B_3
E_2
A_4
B_4
C_6
C_5
A_2
2
B_5
B_6
E_1
4
E_3
C_4
6
3
C_3
A_1
B_2
A_5
C_1

pairs of triangles. Through every center of perspective there pass, therefore, 3 of the 15 sides of the Clebschian hexagon. Besides there are 15 secondary points in which the 15 sides of the hexagon meet each other only by twos. Two centers of perspective lie on each side of the hexagon. If we form a triangle out of any 3 vertices of the hexagon, then the 6 centers of perspective situated on the 3 sides of this triangle, lie on a conic section F. The 15 secondary points can be arranged as 5 triangles in such a way that every side of each triangle is also a side of the Clebschian hexagon. Then the 6 vertices of any 2 of these 5 triangles lie on a conic section Φ.

The connection between the 6 dihedron-points 1, 2, . . . , 6 in Fig. I and the Clebschian hexagon (see Fig. II, where the points corresponding to those in Fig. I are denoted by the same numbers or letters) is now simply this :

1. The dihedron-points 1–6 constitute 6 centers of perspective lying on a conic section F of a Clebschian hexagon whose vertices C_1, C_2, . . . , C_6 are obtained as the points of intersection of the lines joining the points :

2, 3 and 4, 5 : C_1, 2, 3 and 4, 6 : C_4,
3, 1 and 5, 6 : C_2, 3, 1 and 5, 4 : C_5,
1, 2 and 6, 4 : C_3, 1, 2 and 6, 5 : C_6.

2. The dihedron-points 1–6 constitute 6 secondary points lying on a conic section Φ of a Clebschian hexagon whose vertices B_1, B_2, . . . , B_6* are obtained as the points of intersection of the lines joining the points

1, 5 and 2, 6 : B_1, 1, 6 and 3, 5 : B_4,
2, 6 and 3, 4 : B_2, 2, 4 and 1, 6 : B_5,
3, 4 and 1, 5 : B_3, 3, 5 and 2, 4 : B_6.

The configuration of our points 1, 2, . . . , 6 is identical with that special case of two triply perspective triangles enumerated by Hess under 2 (l. c., p. 188), as can be demonstrated by referring to Hess's triangle of reference. Hess himself has shown only (l. c., p. 209) how this figure is connected with the Clebschian hexagon B_1, . . . , B_6. The connection with the hexagon C_1, . . . , C_6 follows without much difficulty by a closer study of the Clebschian configuration. The relation between the two Clebschian hexagons C_1, . . . C_6 and B_1, . . . B_6 is given by the following theorem : *Such* 6 *secondary points as lie*

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* In Fig. II the sides of the Clebschian hexagon C_1, . . . , C_6 are represented by continuous lines, those belonging to the hexagon B_1, . . . , B_6 by broken lines, except the 3 lines B_1C_1, B_2C_2, B_3C_3 which are common to both. The two circumscribed triangles $A_1A_2A_3$ and $A_4A_5A_6$ are given by dotted lines.

on 3 *sides of a Clebschian hexagon intersecting in a center of perspective, form again a Clebschian hexagon.*

Finally it may be remarked that the 6 vertices A_1, A_2, . . . , A_6 of the two triangles circumscribed to the conic section in Moebius figure (see Fig. I) constitute 6 centers of perspective lying on a conic section F of a Clebschian hexagon (see Fig. II). That these points lie on a conic section, has already been found by Moebius. We now see, moreover, that these points constitute again a system of real dihedron-points of the same character as the original points 1, 2, . . . , 6.

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